CANONICAL METRICS OF COMMUTING MAPS

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ABSTRACT. Let $\varphi : X \to X$ be a map on an projective variety. It is known that whenever $\varphi^* : \text{Pic}(X) \otimes \mathbb{R} \to \text{Pic}(X) \otimes \mathbb{R}$ has an eigenvalue $\alpha > 1$, we can build a canonical measure, a canonical height and a canonical metric associated to φ. In the present work, we establish the following fact: if two **commuting maps** $\varphi, \psi : X \to X$ satisfy these conditions, for eigenvalues α and β and the same eigenvector $\mathcal{L}$, then the canonical metric, the canonical measure, and the canonical height associated to both maps, are identical.

1. INTRODUCTION

Let X be a projective variety defined over a number field K. Suppose that $\varphi : X \to X$ is a map on X, also defined over K. Assume that we can find an ample line bundle $\mathcal{L}$ on X and a number $\alpha > 1$, such that $\mathcal{L}^\alpha \cong \varphi^*\mathcal{L}$. Under this conditions, we can build the canonical height $\hat{h}_\varphi$ ([9] theorem 1.1) and the canonical measure $d\mu_\varphi$ ([20] proposition 3.1.4) associated to φ and $\mathcal{L}$. They satisfy nice properties with respect to the map φ, for example we have $\hat{h}_\varphi \circ \varphi = \deg(\varphi)\hat{h}_\varphi$ and $\varphi_*\mu_\varphi = \mu_\varphi$. Sometimes it happens that a whole set of maps are associated to the same canonical height function and measure. As our first example consider the collection of maps $\phi_k : \mathbb{P}^1_{\bar{K}} \to \mathbb{P}^1_{\bar{K}}$ on the Riemann Sphere, where ϕ_k is defined as $\phi_k(t) = t^k$. The line bundle $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^1$ satisfies the isomorphism $\phi_k^*\mathcal{L} \cong \mathcal{L}^k$. If one builds the canonical height and measure associated to ϕ_k and $\mathcal{O}(1)$, one obtains:

(i) All ϕ_k have the same canonical height namely, the naive height h_{nv} on $\mathbb{P}^1_{\mathbb{Q}}$. The naive height $h_{nv}(P)$ is a refined idea of the function $\sup\{|a_0|, |a_1|\}$, measuring the computational complexity of the projective point $P = (a_0 : a_1)$. For a precise definition see later definition 2.10.

(ii) All ϕ_k have the same canonical measure, that is, the Haar measure $d\theta$ on the unit circle S^1.

Similar properties are fulfilled by the collection of maps $[n] : E \to E$, representing multiplication by n on an elliptic curve E defined over K. If $\mathcal{L}$ is an ample symmetric line bundle on E, we have the isomorphism $[n]^*\mathcal{L} \cong \mathcal{L}^{n^2}$, along with the properties:

(i) All maps $[n]$ share the same canonical height, that is, the Neron-Tate height $\hat{h}_E$ on E. In fact this will be our definition (2.11) of the Neron-Tate height on E. For many other interesting properties we refer to B-4 in [8].

(ii) All maps $[n]$ have the same canonical measure, that is, the Haar measure $i/2Im(\tau)dz \wedge d\bar{z}$ on $E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

We observe that any two maps in each collection commute for the composition of maps. Besides, the line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$, suitable to make everything work, is the same within each collection. The present work establish the general fact:

Proposition 1.1. *Let X be a projective variety defined over a number field K. Suppose that two maps $\varphi, \psi : X \to X$ commute ($\varphi \circ \psi = \psi \circ \varphi$) and satisfy the following property: For some ample line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ and real numbers $\alpha, \beta > 1$, we have $\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$, then we have $\hat{h}_\varphi = \hat{h}_\psi = \hat{h}_{\varphi \circ \psi}$ and $d\mu_\varphi = d\mu_\psi = d\mu_{\varphi \circ \psi}$.*

This result is known in dimension one, a proof can found for example in [6]. Also it is a well known fact [8], that commuting maps in a projective variety must share the same canonical height. The main feature of the present work it is to obtain all this results from the equality of the canonical metrics. Given a ample line bundle $\mathcal{L}$ on X, it was an original idea of Arakelov [1] to put metrics on $\mathcal{L}_\sigma = \mathcal{L} \otimes_\sigma \mathbb{C}$ over all places σ of K at infinity. This gave rise to heights as intersection numbers and curvature forms at infinity. In was then an idea of Zhang [19] to look for suitable metrics at all places v of K. In presence of the dynamics $\varphi : X \to X$, the line bundle $\mathcal{L}$ on X can be endowed [19] with very special metrics $\|.\|_{\varphi,v}$ on $\mathcal{L}_v$ that satisfy the functional equation

$$\|.\|_{\varphi,v} = (\phi^*\varphi^*\|.\|_{\varphi,v})^{1/\alpha},$$

whenever we have an isomorphism $\phi : \mathcal{L}^\alpha \xrightarrow{\sim} \varphi^*\mathcal{L}$. The canonical height and the canonical metric will be defined (definitions 2.6 and 2.9) depending only on the metric $\|.\|_\varphi$. The equality of canonical heights and measure for commuting maps is a consequence of the following proposition:

Proposition 1.2. *Suppose that two maps $\varphi, \psi : X \to X$ commute, and for some ample line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ we have $\varphi^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$ for some numbers $\alpha, \beta > 1$, then $\|.\|_\varphi = \|.\|_\psi$.*

Towards the end of the paper we discuss maps on $\mathbb{P}^1$ arising as projections of maps on elliptic curves with complex multiplication. We study ramification points and present examples of commuting maps on the Riemann sphere.

2. Canonical heights and canonical measures

2.1. **Canonical metrics.** Consider the projective variety X defined over a number field K, a map $\varphi : X \to X$ defined over K, and an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ such that $\phi : \mathcal{L}^\alpha \xrightarrow{\sim} \varphi^* \mathcal{L}$ for some $\alpha > 1$. This situation will be called [20] a polarized dynamical system $(X, \varphi, \mathcal{L}, \alpha)$ on X defined over K.
Assume that for every place v of K we have chosen a continuous and bounded metric $\|.\|_v$ on each fibre of $\mathcal{L}_v = \mathcal{L} \otimes_K K_v$. The following theorem is proposition 2.2 in [19]:

Theorem 2.1. *The sequence defined recurrently by $\|.\|_{v,1} = \|.\|_v$ and $\|.\|_{v,n} = (\phi^* \varphi^* \|.\|_{v,n-1})^{1/\alpha}$ for $n > 1$, converge uniformly on $X(\bar{K}_v)$ to a metric $\|.\|_{v,\varphi}$ (independent of the choice of $\|.\|_{v,1}$) on $\mathcal{L}_v$ which satisfies the equation $\|.\|_{\varphi,v} = (\phi^* \varphi^* \|.\|_{\varphi,v})^{1/\alpha}$.*

Proof. Denote by h the continuous function $\log \frac{\|.\|_2}{\|.\|_1}$ on $X(\bar{K}_v)$. Then

$$\log \|.\|_n = \log \|.\|_1 + \sum_{k=0}^{n-2} (\frac{1}{\alpha} \phi^* \varphi^*)^k h.$$

Since $\|(\frac{1}{\alpha} \phi^* \varphi^*)^k h\|_{sup} \leq (\frac{1}{\alpha})^k \|h\|_{sup}$, it follows that the series given by the expression $\sum_{k=0}^{\infty} (\frac{1}{\alpha} \phi^* \varphi^*)^k h$, converges absolutely to a bounded and continuous function h^v on $X(\bar{K}_v)$. Let $\|.\|_{\varphi,v} = \|.\|_1 \exp(h^v)$, then $\|.\|_n$ converges uniformly to $\|.\|_{\varphi,v}$ and its not hard to check that $\|.\|_{\varphi,v}$ satisfies

$$\|.\|_{\varphi,v} = (\phi^* \varphi^* \|.\|_{\varphi,v})^{1/\alpha},$$

which was the result we wanted to prove. $\square$

Definition 2.2. *The metric $\|.\|_{v,\varphi}$ is called the canonical metric on $\mathcal{L}_v$ relative to the map φ.*

Example 2.3. *Consider the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n_{\mathbb{Q}}$ and the rational map $\phi_k : \mathbb{P}^n_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$ given by the expression $\phi(T_0 : ... : T_n) =$*

$(T_0^k : ... : T_n^k)$. *The Fubini-Study metric*

$$\|(\lambda_0 T_0 + ... + \lambda_n T_n)(a_0 : ... : a_n)\|_{FS} = \frac{|\sum \lambda_i a_i|}{\sqrt{\sum_i a_i^2}}$$

is a smooth metric on $\mathcal{L}_{\mathbb{C}}$. If we take $\|.\|_1 = \|.\|_{FS}$ as our metric at infinity, the limit metric we obtain is

$$\|(\lambda_0 T_0 + ... + \lambda_n T_n)(a_0 : ... : a_n)\|_{nv} = \frac{|\sum \lambda_i a_i|}{\sup_i(|a_i|)}.$$

Example 2.4. *Suppose that $X = E$ is an elliptic curve and assume that $[n] : E \to E$ is denoting the multiplication by n on E. As a consequence of the theorem of the cube, the ample symmetric line bundle $\mathcal{L}$ on E satisfies $\phi : [n]^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^{n^2}$. The canonical metric is the metric of the cube discussed in [11] and suitable to make ϕ and isomorphism of metrized line bundles.*

The following proposition relates the canonical metrics associated to commuting maps. It represents the main result of this paper.

Proposition 2.5. *Let $(X, \varphi, \mathcal{L}, \alpha)$ and $(X, \psi, \mathcal{L}, \beta)$ be two polarized systems on X defined over K. Suppose that the maps φ and ψ satisfy $\varphi \circ \psi = \psi \circ \varphi$, then $\|.\|_\varphi = \|.\|_\psi$.*

Proof. The key idea is that the canonical metric associate to a morphism does not depend on the metric we start the iteration with. Let $s \in \Gamma(X, \mathcal{L})$ be a non-zero section of $\mathcal{L}$. We are going to consider two metrics $\|.\|_{v,1} = \|.\|_\varphi$ and $\|.\|'_{v,1} = \|.\|_\psi$ on the line bundle $\mathcal{L}$. By our definition of canonical metric for φ, we can start with $\|.\|'_{v,1}$ and obtain $\|s(x)\|_\varphi = \lim_k \|s(\varphi^k(x)\|_\psi^{1/\alpha^k}$, but also by our definition of canonical metric for ψ starting with $\|.\|_{v,1} = \|.\|_\varphi$ we get $\|s(x)\|_\psi = \lim_l \|s(\varphi^l(x))\|_{v,1}^{1/\beta^l}$. So using the uniform convergence and the commutativity of the maps,

$$\|s(x)\|_\varphi = \lim_{k,l} \|s(\varphi^k \circ \psi^l(x))\|_{v,1}^{1/\alpha^k \beta^l}$$

$$= \lim_{l,k} \|s(\psi^l \circ \varphi^k(x))\|_{v,1}^{1/\beta^l \alpha^k} = \|s(x)\|_\psi,$$

which was the result we wanted to prove. $\square$

2.2. **Canonical measures.** Let X be a n-dimensional projective variety defined over a number field K and suppose that $(X, \varphi, \mathcal{L}, \alpha)$ is a polarized dynamical system defined over K. let v be a place of K over infinity. We can consider the morphism $\varphi \otimes v : X_v \to X_v$ on the complex variety X_v. Associated to φ and v we also have the

canonical metric $\|.\|_{\varphi,v}$ and therefore the distribution $c_1(\mathcal{L}, \|.\|_{\varphi,v}) = \frac{1}{(\pi i)} \partial \overline{\partial} \log \|s_1(P)\|_{\varphi,v}$ analogous to the first Chern form of $(\mathcal{L}, \|.\|_{\varphi,v})$. It can be proved that $c_1(\mathcal{L}, \|.\|_{\varphi,v})$ is a positive current in the sense of Lelong, and following [5] we can define the n-product

$$c_1(\mathcal{L}, \|.\|_{\varphi,v})^n = c_1(\mathcal{L}, \|.\|_{\varphi,v})...c_1(\mathcal{L}, \|.\|_{\varphi,v}),$$

which represents a measure on X_v.

Definition 2.6. *The measure* $d\mu_\varphi = c_1(\mathcal{L}_v, \|.\|_{\varphi,v})^n/\mu(X)$, *is called the canonical measure associated to* φ *and* v. *Once we have fixed* $\mathcal{L}$, *it depends only on the metric* $\|.\|_{\varphi,v}$.

Example 2.7. *Consider the rational map* $\phi_k : \mathbb{P}^n_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$ *given by the formula* $\phi_k(T_0 : ... : T_n) = (T_0^k : ... : T_n^k)$, *the canonical measure* $d\mu_{\phi_k}$ *is the normalize Haar measure on the n-torus* $S^1 \times ... \times S^1$.

Example 2.8. *Let* E *be an elliptic curve,* $\mathcal{L}$ *a symmetric line bundle on* E *and the map* $[n] : E \to E$. *The canonical measure associated to* $(E, [n], \mathcal{L}, [n]^2)$ *can be proved to be* [11] *the normalized Haar measure on* E.

2.3. **Canonical heights as intersection numbers.** For a regular projective variety X of dimension n, defined over a field K, the classical theory of intersection ([7],[14]) defines the intersection $c_1(\mathcal{L}_1)...c_1(\mathcal{L}_n)$ of the classes $c_1(\mathcal{L}_i)$ associated to line bundles $\mathcal{L}_i$ on X, when $0 < i \le n$. For the purpose of defining the arithmetic intersection, we want to assume that X is an arithmetic variety of dimension $n+1$, that is, given a number field K, there exist a map $f : X \to \mathrm{Spec}(\mathcal{O}_K)$, flat and of finite type over $\mathrm{Spec}(\mathcal{O}_K)$. We can define (See for example [4], [3], [16], [1], [17] or [18]) the arithmetic intersection number $\tilde{c}_1(\mathcal{L}_1)...\tilde{c}_1(\mathcal{L}_{n+1})$ of the classes $\tilde{c}_1(\mathcal{L}_i)$ of hermitian line bundles $\tilde{\mathcal{L}}_i = (\mathcal{L}_i, \|.\|)$ on X. The fact that $\tilde{\mathcal{L}}_i$ are hermitian line bundles for $i = 1..n + 1$, means that each line bundle $\mathcal{L}_i$ on X is equipped with a hermitian metric $\|.\|_{v,i}$ over $X_v = X \otimes_K \mathrm{Spec}\,\mathcal{O}_{K_v}$ for each place v at infinity. The numbers $\tilde{c}_1(\mathcal{L}_1)...\tilde{c}_1(\mathcal{L}_j)$ prove to be the appropriated theory of intersection in the particular case of arithmetic varieties, adding places over infinity allows us to recover the desirable properties of the classical intersection numbers of varieties over fields.

The last step in the theory of intersection is actually the one that plays the more important role in our definition of the canonical height associated to a morphism. Suppose that X is a regular variety of dimension n and $(\mathcal{L}_i, \|.\|_i)_v$ $(i = 1, .., p + 1)$ are metrized line bundles on X. Assume also that the $\mathcal{L}_i$ are been equipped with semipositive metrics over all places v (not just at infinity as before) in the

sense of [19]. Such line bundles are called adelic metrized line bundles and following [19], we can define the adelic intersection number $\hat{c}_1(\mathcal{L}_1|Y)...\hat{c}_1(\mathcal{L}_{p+1}|Y)$ over a $p-$cycle $Y \subset X$. The adelic intersection number is in fact a limit of classical numbers $\tilde{c}_1(\mathcal{L}_1)...\tilde{c}_1(\mathcal{L}_{p+1})$ once the notion of converge is established. The numbers $\hat{c}_1(\mathcal{L}_1|Y)...\hat{c}_1(\mathcal{L}_{p+1}|Y)$ satisfy again nice properties, they are multilinear in each of the $\mathcal{L}_i$ and satisfy $\hat{c}_1(f^*\mathcal{L}_1|Y)...\hat{c}_1(f^*\mathcal{L}_{p+1}|Y) = \hat{c}_1(\mathcal{L}_1|f(Y))...\hat{c}_1(\mathcal{L}_{p+1}|f(Y))$, whenever we have a map $f : X \to X$. We are interested in a particular case of this situation. Suppose that we are in the presence of a polarized dynamical system $(X, \varphi, \mathcal{L}, \alpha)$, in this situation the canonical metric $\|.\|_\varphi$ of 2.1 represent a semipositive metric on $\mathcal{L}$, (again we refer to [19]) and we can define the canonical height associated to $(\mathcal{L}, \|.\|_\varphi)$.

Definition 2.9. *The canonical height $\hat{h}_\varphi(Y)$ of a $p-$cycle Y in X is defined as*

$$\hat{h}_\varphi(Y) = \frac{\hat{c}_1(\mathcal{L}|Y)^{p+1}}{(\dim(Y)+1)c_1(\mathcal{L}|Y)^p}.$$

It depends only on $(\mathcal{L}, \|.\|_\varphi)$, where $\|.\|_\varphi$ is actually representing a collection of canonical metrics over all places of K. An important particular case of canonical height will be the canonical height $\hat{h}_\varphi(P)$ of a point in $P \in X$.

Example 2.10. *Consider the map $\phi_k : \mathbb{P}_\mathbb{Q}^n \to \mathbb{P}_\mathbb{Q}^n$ given by the formula $\phi_k(T_0 : ... : T_n) = (T_0^k : ... : T_n^k)$, the canonical height associated to ϕ_k is called the naive height h_{nv} on $\mathbb{P}^n$. If $P = [t_0 : ... : t_n]$ is a point in $\mathbb{P}^n$ the naive height is*

$$h_{nv}([t_0 : ... : t_n]) = \frac{1}{[K : \mathbb{Q}]} \log \prod_{places\ v\ of\ K} \sup(|t_0|_v, ..., |t_n|_v)^{\mathrm{N}_v},$$

where $\mathrm{N}_v = [K_v : \mathbb{Q}_w]$ and w is the place of $\mathbb{Q}$ such that $v \mid w$.

Definition 2.11. *Let E be an elliptic curve and $\mathcal{L}$ an ample symmetric line bundle on E. The canonical height associated to $[n] : E \to E$ and $\mathcal{L}$ is called the Neron-Tate height $\hat{h}_E$ on E. The fact that this is independent of n, will be a consequence of proposition 2.12.*

The collection of maps $\{\phi_k\}_k$ on $\mathbb{P}^n$ and the collection $\{[n]\}_n$ on a given elliptic curve E, share two important properties, the maps within each collection commute, and share the same canonical height and canonical measure. The following proposition establishes a general fact about canonical heights and canonical measures of commuting maps on a projective variety X.

Proposition 2.12. *Let $(X, \varphi, \mathcal{L}, \alpha)$ and $(X, \psi, \mathcal{L}, \beta)$ be two polarized systems on X defined over K. Suppose that the maps φ and ψ satisfy $\varphi \circ \psi = \psi \circ \varphi$, then $\hat{h}_\varphi = \hat{h}_\psi = \hat{h}_{\varphi \circ \psi}$ and $d\mu_\varphi = d\mu_\psi = d\mu_{\varphi \circ \psi}$.*

Proof. This is a consequence of our definitions of canonical measure 2.6, canonical height 2.9 and proposition 2.5. $\qquad\square$

Corollary 2.13. *Suppose that two maps $\varphi, \psi : \mathbb{P}^1 \to \mathbb{P}^1$, satisfy the hypothesis of the previous proposition, then the two maps have the same Julia set.*

Proof. The Julia set of a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is nothing but the closure in $\mathbb{P}^1$ of the set of repelling periodic points. For details we refer to definition 2.2 in [12]. Now, the corollary is a consequence of proposition 2.12 and proposition 7.2 in [12]. $\qquad\square$

3. ELLIPTIC CURVES AND EXAMPLES

This section illustrates examples of commuting maps on $\mathbb{P}^1$. They all share one thing in common: being induced in some sense by endomorphisms on elliptic curves.

Proposition 3.1. *Consider an elliptic curve $E = \mathbb{C}/1\mathbb{Z} + \tau\mathbb{Z}$ given by Weierstrass equation $y^2 = G(x)$. Suppose that E admits multiplication by the algebraic number λ, then multiplication by λ in E induces, as quotient by the action of $[-1]$, a map $\varphi_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$. Besides, we have:*

(i) *$\hat{h}_E(x, y) = \hat{h}_\lambda(x)$ for any point $P = (x, y)$ on E.*
(ii) *The canonical measure on $\mathbb{P}^1$ associated to φ_λ is*

$$d\mu_{\varphi_\lambda} = \frac{i dz \wedge d\bar{z}}{2 Im(\tau)|G(z)|}.$$

Proof. The first part is a classical fact of the theory of elliptic functions and complex multiplication. There exist polynomials $P(z)$ and $Q(z)$ where $\deg(P) = \deg(Q)+1 = N(\lambda)$ such that $\wp(\lambda z) = P(\wp(z))/Q(\wp(z))$, where $\wp$ is denoting the Weierstrass $\wp$−function. Suppose that we call π the quotient map from $E \to \mathbb{P}^1$, we have a commutative diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{\ \lambda\ } & E \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{P}^1 & \xrightarrow{\ \varphi_\lambda\ } & \mathbb{P}^1
\end{array}
$$

Now, consider the line bundle $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^1$, we have $\varphi_\lambda^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{N(\lambda)}$ and equally for the ample symmetric line bundle $\pi^* \mathcal{L}$ on E. Therefore, it make sense to talk about canonical heights associated to $\varphi_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$

and $\lambda : E \to E$. The number λ lies in an imaginary quadratic extension of $\mathbb{Q}$, so we also have a commutative diagram:

$$
\begin{array}{ccccc}
E & \xrightarrow{\ \lambda\ } & E & \xrightarrow{\ \bar{\lambda}\ } & E \\
\pi \downarrow & & \pi \downarrow & & \pi \downarrow \\
\mathbb{P}^1 & \xrightarrow{\ \varphi_\lambda\ } & \mathbb{P}^1 & \xrightarrow{\ \varphi_{\bar\lambda}\ } & \mathbb{P}^1
\end{array}
$$

So, the two maps φ_λ and $\varphi_{\bar\lambda}$ commute. After 2.12 the canonical height associated to multiplication by λ on E is the same as the canonical height associated to multiplication by $N(\lambda)$, that is the Neron-Tate height on E. Take $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^1$ and P a point on E, the intersection numbers satisfy a projection formula

$$
\hat{c}_1(\pi^*\mathcal{L}|P) = \hat{c}_1(\mathcal{L}|\pi(P)) \qquad c_1(\pi^*\mathcal{L}|P) = c_1(\mathcal{L}|\pi(P)).
$$

This gives (i) after definition 2.9. For (ii) consider the Haar measure $i/2dz \wedge d\bar{z}$ on E, normalized by $Im(\tau)$. If $\wp$ denote the Weierstrass function and $\omega = \wp(z)$, we have

$$
\frac{id\omega \wedge d\bar{\omega}}{2Im(\tau)} = \frac{idz \wedge d\bar{z}}{2|\wp'(z)|^2 Im(\tau)} = \frac{idz \wedge d\bar{z}}{2|y^2|Im(\tau)} = \frac{idz \wedge d\bar{z}}{2|G|Im(\tau)}.
$$

which gives the result we wanted to prove. $\square$

Remark 3.2. *If the elliptic curve E admits multiplication by the numbers λ and δ, then $\varphi_\lambda \circ \varphi_\delta = \varphi_\delta \circ \varphi_\lambda$.*

Example 3.3. *Consider an elliptic curve E given by Weierstrass equation $E : y^2 = G(x)$. For $\lambda = 2$ we have*

$$
\varphi_2(z) = \frac{(G'(z))^2 - 8zG(z)}{4G(z)}.
$$

Example 3.4. *Let's consider some examples of elliptic curves with complex multiplication:*

The elliptic curve $E_1 : y^2 = x^3 + x$ admits multiplication by $\mathbb{Z}[i]$. The multiplication by i morphism can be written in x, y coordinates as $[i](x, y) = (-x, iy)$. The two maps

$$
\varphi_{1+i}(z) = \frac{1}{(1+i)^2}\frac{z^2+1}{z} \qquad \varphi_{1-i}(z) = -\frac{1}{(1+i)^2}\frac{z^2+1}{z}
$$

commute, and their composition satisfies

$$
\varphi_{1+i}(\varphi_{1-i}(z)) = \varphi_{1-i}(\varphi_{1+i}(z)) = \varphi_2(z) = \frac{z^4 - 2z^2 + 1}{4(z^3 + z)}.
$$

The canonical height and measure are:

$$\hat{h}(z) = h_{E_1}(z, \pm\sqrt{z^3 + z}) \qquad d\mu(z) = \frac{idz \wedge d\bar{z}}{2|z^3 + z|}$$

Other examples of maps attached to E_1 are

$$\varphi_{1+2i}(z) = \frac{(-3 - 4i)z(z^2 + 1 + 2i)^2}{(5z^2 + 1 - 2i)^2} \qquad \varphi_{1-2i}(z) = \frac{(3 + 4i)z(z^2 + 1 + 2i)^2}{(5z^2 + 1 - 2i)^2}$$

$$\varphi_{2+i}(z) = \frac{(3 - 4i)z(z^2 + 1 - 2i)^2}{(5z^2 + 1 + 2i)^2} \qquad \varphi_{2-i}(z) = \frac{(-3 + 4i)z(z^2 + 1 - 2i)^2}{(5z^2 + 1 + 2i)^2}.$$

The curve $E_2 : y^2 = x^3 + 1$ admits multiplication by the ring $\mathbb{Z}[\rho]$ where $\rho = (\sqrt{-3} + 1)/2$. The multiplication by ρ can be expressed in x, y coordinates as $[\rho](x, y) = (\rho x, y)$. An example of commuting maps coming from E_2 is

$$\varphi_{\sqrt{-3}}(z) = \frac{-(z^3 + 4)}{3z^2} \qquad \varphi_{\sqrt{-3}\rho}(z) = \frac{-\rho(z^3 + 4)}{3z^2}$$

$$\varphi_{\sqrt{-3}} \circ \varphi_{\sqrt{-3}\rho}(z) = \varphi_\varepsilon(z) = \frac{(z^9 - 96z^6 + 48z^3 + 64)}{9\rho z^2(z^3 + 4)^2},$$

where $\varepsilon = (-3\sqrt{-3} + 3)/2$. The canonical measure associated to the three maps is

$$d\mu_{E_2}(z) = \frac{\sqrt{3}idz \wedge d\bar{z}}{3|z^3 + 1|}.$$

To have an idea of the ramification points and indexes of the maps φ_λ, we proof the following lemma:

Lemma 3.5. *A ramification point for φ_λ belongs to the image by π of the 2-torsion points on E.*

Proof. To see this, suppose that $\varphi_\lambda^{-1}(\pi(P)) = \{\pi(Q)|\lambda Q = P\}$ has cardinal strictly smaller than $N(\lambda)$. Then there exist two points $\pi(Q) \neq \pi(-Q)$ inside the set $\varphi_\lambda^{-1}(P)$, such that $\lambda Q = -\lambda Q$ and consequently $2\lambda Q = 2P = 0$. □

Let's see some examples of the different ramifications that a map φ_λ may have. Let d be a positive square free integer. Assume that the elliptic curve $\mathbb{C}/Z + \sqrt{-d}Z$, admits multiplication by $\lambda = a + b\sqrt{-d}$. Suppose that $P_0 = 0$, $P_1 = 1/2$, $P_2 = 1/2 + \sqrt{-d}/2$ and $P_3 = \sqrt{-d}/2$ denote the 2-torsion points on E and that r_j denotes the amount of pre-images of the point $\pi(P_j)$, that is, the cardinality of the set $\varphi_\lambda^{-1}(\pi(P_j))$. Under the conditions previously described, we can observe for example that for $\lambda = 2$, the points in $\varphi_2^{-1}(\pi(P_0))$ are not ramification points

of φ_2. On the other hand for the multiplication by $\lambda = 1 + 2i$ on E_1, all points in $\varphi_{1+2i}^{-1}(\pi(P_0)) \cup \varphi_{1+2i}^{-1}(\pi(P_1)) \cup \varphi_{1+2i}^{-1}(\pi(P_2)) \cup \varphi_{1+2i}^{-1}(\pi(P_3))$ are ramification points of φ_{1+2i}. The following table summarize the results:

$\lambda = a + b\sqrt{-d},\ N(\lambda)$	$r_j, j = 0, 2$	$r_j, j = 1, 3$
$a + bd \equiv 1 mod(2)$	$r_0 = (N(\lambda) + 1)/2$	$r_1 = (N(\lambda) + 1)/2$
	$r_2 = (N(\lambda) + 1)/2$	$r_3 = (N(\lambda) + 1)/2$
$a \equiv b \equiv 0 mod(2),\ N(\lambda) > 4$	$r_0 = N(\lambda)/2 + 2$	$r_1 = N(\lambda)/2$
	$r_2 = N(\lambda)/2$	$r_3 = N(\lambda)/2$
$a \equiv b \equiv 0 mod(2),\ N(\lambda) = 4$	$r_0 = 4$	$r_1 = N(\lambda)/2$
	$r_2 = N(\lambda)/2$	$r_3 = N(\lambda)/2$
$a \equiv bd \equiv 1 mod(2),\ N(\lambda) = 2$	$r_0 = 1$	$r_1 = N(\lambda)$
	$r_2 = 1$	$r_3 = N(\lambda)$
$a \equiv bd \equiv 1 mod(2),\ N(\lambda) > 2$	$r_0 = N(\lambda)/2$	$r_1 = N(\lambda)/2 + 1$
	$r_2 = N(\lambda)/2$	$r_3 = N(\lambda)/2 + 1$

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